UDC 539.3

ON A DYNAMIC PROBLEM FOR AN INFINITE CYLINDER

PMM Vol. 39, № 3, 1975, pp. 555-559
V. V. KALINCHUK
(Rostov-on-Don)
(Received December 4, 1973)

The radial vibrations of an infinite elastic circular cylinder of unit radius under the effect of a fitted stiff strap of width $2a$ performing oscillations at a frequency ω are considered. The solvability of this problem and the uniqueness of its solution at all the frequencies under consideration in this paper are given a foundation. A method is proposed for constructing the approximate formulas describing the state of stress under a stamp and the behavior of the cylinder outside the stamp with given accuracy. Results are given of a numerical analysis of the wave processes under the stamp and on the free surface of the cylinder for a flat stamp.

1. Using the principle of limiting absorption [1, 2], the problem is reduced to the solution of the integral equation

$$\int_{-a}^{a} k(x-t)\, q(t)\, dt = 2\pi f(x), \qquad |x| \leqslant a \tag{1.1}$$

$$k(t) = \int_{\sigma} K(u)\, e^{iut}\, du \tag{1.2}$$

$$K(u) = -\varkappa^2\sigma_1 \left[(u^2 - {}^1/_2\, \varkappa^2)^2 I(\sigma_1) - u^2\sigma_1\sigma_2 I(\sigma_2) + 2\varkappa^2\sigma_1\right]^{-1} \tag{1.3}$$

$$\sigma_1 = (u^2 - \mu\varkappa^2)^{1/2}, \quad \sigma_2 = (u^2 - \varkappa^2)^{1/2}, \quad \varkappa^2 = \rho\omega^2 G^{-1}$$

$$\mu = ({}^1/_2 - \nu)(1-\nu)^{-1}, \quad I(u) = 4I_0(u)\, I_1^{-1}(u)$$

Here $I_0(u)$, $I_1(u)$ are Bessel functions of imaginary argument, ν is the Poisson's ratio, G is the shear modulus, ρ is the material density, $q(x)$ and $f(x)$ are functions characterizing the contact stresses and the displacement of points of the cylinder surface under the stamp, respectively.

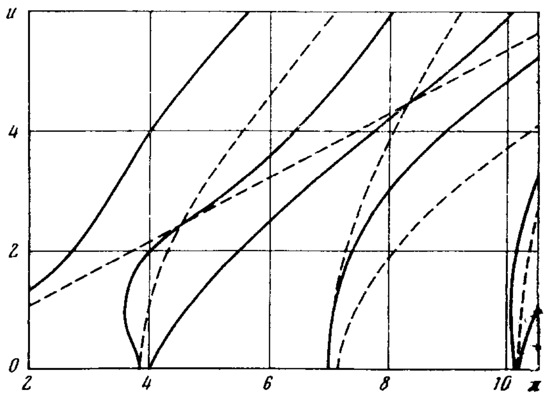

Fig. 1

The function $K(u)$ in (1.3) is analytic in the complex domain, even, real and with a finite number of zeros and poles on the real axis. As $u \to \infty$

$$K(u) = c_1 u^{-1} + c_2 u^{-2} + O(u^{-3}) \tag{1.4}$$

The expansion (1.4) is characteristic for integral equation kernels originating in the investigation of problems with cylindrical contact domains [3]. In the case under consideration

$c_1 = 1 - \nu,\ c_2 = {}^1/_2\,(1 - \nu)\,(3 - 4\nu)$.

The contour σ in (1.2) lies on the real axis and deviates from it only in passing above the negative and below the positive zeros and poles in $K(u)$ in (1.3) [1, 2].

2. Presented in Fig. 1 are neutral curves showing the distribution of the real zeros (dashed lines) and the poles (solid lines) of the kernel $K(u)$ in (1.3) as a function of the value of the parameter $\varkappa$ ($\nu = 0.3$). It is seen that as the frequency increases (in this case the parameter $\varkappa$) , the quantity of zeros m_1 and poles m increase (in the general case $m \geqslant m_1$). Let us note [1] that m_1 and m is the quantity of wave pairs with nondecreasing amplitude under the stamp and on the free cylinder surface, respectively.

It is easy to detect that strict alternation of the zeros and poles is observed (this holds even for values of $\varkappa$ corresponding to intersections of the neutral curves since the function $K(u)$ evidently has just a single zero at these points) for each fixed $\varkappa$ (with the exception of cases of convexity of the neutral curves to the left, which are not examined here).

The solvability of the integral equation (1.1) and the uniqueness of its solution at all the frequencies under consideration follow from the condition of strict alternation of the zeros and poles [1, 2].

3. To construct the approximate solution of the integral equation (1.1), let us replace the kernel $K(u)$ in (1.3) by the function

$$K^*(u) = c_1\,(u^2 + B^2)^{-1/2} \prod_{k=1}^{n} (u^2 - z_k^2)\,(u^2 - \gamma_k^2)^{-1} \tag{3.1}$$

Here $B \gg 1$ is an approximation parameter given in advance, γ_k ($k = 1, 2, \ldots m$) and z_k ($k = 1, 2, \ldots m_1$) are real poles and zeros of the function $K(u)$ in (1.3). The remaining $\gamma_k = iA$ ($k = m + 1, m + 2, \ldots n$), $A > 0$ is an additional approximation parameter which must be introduced as dictated by the use of a Bernshtein polynomial as an approximating function [4], and z_k ($\operatorname{Im} z_k > 0$, $k = m_1 + 1, m_1 + 2, \ldots n$) are complex numbers which satisfy the condition of the least deviation of $K^*(u)$ in (3.1) from $K(u)$ in (1.3) on the real axis together with the parameter A [5]. An error estimate of the approximate solution of the integral equation is given in [5], where a form of the approximating function similar to (3.1) is used.

4. Let us examine the case $f(x) = \exp(i\eta x)$. The solution of (1.1) is

$$q(x) = -\frac{1}{2\pi i}\int_\sigma \left\langle \left[\frac{e^{-ia\eta}}{K_+(\eta)\,(u+\eta)} - \sum_{k=1}^{n} \frac{a_k^+}{2\,(u - z_k)}\right] \frac{e^{-iu(a+x)}}{K_+(u)} - \right. \tag{4.1}$$

$$\left.\left[\frac{e^{ia\eta}}{K_-(\eta)\,(u+\eta)} + \sum_{k=1}^{n} \frac{a_k^-}{2\,(u + z_k)}\right] \frac{e^{iu(a-x)}}{K_-(u)} \right\rangle du + O\left(e^{-2Ba}\right)$$

$$K_\pm(u) = \sqrt{c_1}\,(B \mp iu)^{-1/2} \prod_{k=1}^{n} (z_k \pm u)\,(\gamma_k \pm u)^{-1}$$

$$a_k^\pm = r_k \sum_{s=1}^{n} \left[B_{ks}^- \alpha^-(-z_s) \pm B_{ks}^+ \alpha^+(-z_s)\right]$$

$$r_k = e^{2aiz_k} K_+(z_k)\,[K_+'(-z_k)]^{-1}$$

$$\alpha^\pm(z) = e^{ia\eta} K_-^{-1}(\eta)\,(z - \eta)^{-1} \pm e^{-ia\eta} K_+^{-1}(\eta)\,(z + \eta)^{-1}$$

Here $B_{ks}^{\pm}$ are elements of the matrix inverse to a matrix of the form $\| \delta_{ik} \pm (z_i + z_k)^{-1} r_k \|_1^n$, and δ_{ik} is the Kronecker symbol.

The behavior of the cylinder surface outside the stamp is described by the functions

$$\varphi^{\pm}(x) = e^{i\eta x} + \frac{1}{2\pi i} \int_{-\infty - i\varepsilon}^{\infty + i\varepsilon} K_+(u)\, e^{iu(a \mp x)} \Big[e^{\pm ia\eta} K_{\mp}^{-1}(\eta)(u + \eta)^{-1} + \tag{4.2}$$

$$\frac{1}{2} \sum_{k=1}^{n} a_k^{\mp} (u + z_k)^{-1} \Big] du + O(e^{-2Ba}), \quad \pm x > a$$

Reducing the right sides of (4.1) and (4.2) to a form convenient for the application of operational calculus formulas, and using these latter, we obtain

$$q(x) = e^{i\eta x} K^{-1}(\eta) [T^+(\eta) + T^-(-\eta) - 1] + A^+ e^{-B(a+x)} [\pi(a + x)]^{-1/2} + \tag{4.3}$$

$$A^- e^{-B(a-x)} [\pi(a - x)]^{-1/2} + \sum_{k=1}^{n} [N_k^+ e^{iz_k(a+x)} + N_k^- e^{iz_k(a-x)}] +$$

$$\frac{1}{2} \sum_{k=1}^{n} K_+^{-1}(z_k) [a_k^+ (1 - T^+(-z_k))\, e^{-iz_k(a+x)} -$$

$$a_k^- (1 - T^-(-z_k))\, e^{-iz_k(a-x)}] + O(e^{-2Ba}), \quad |x| \leqslant a$$

$$A^{\pm} = \frac{e^{\mp ia\eta}}{K(\eta)\sqrt{B \pm i\eta}} + \sum_{k=1}^{n} \frac{S^{\pm}(z_k)}{K_+'(-z_k)\sqrt{B + iz_k}} \mp$$

$$\frac{1}{2} \sum_{k=1}^{n} \frac{a_k^{\pm}}{K_+(z_k)\sqrt{B - iz_k}}$$

$$\varphi^{\pm}(x) = e^{i\eta x} [1 - T_0^{\pm}(\pm\eta)] + \sum_{k=1}^{m} M_k^{\pm} e^{i\gamma_k(-a \pm x)} + X^{\pm}(x), \quad \pm x > a \tag{4.4}$$

$$X^{\pm}(x) = \frac{i(-1)^l}{l!} \frac{\partial^l}{\partial A^l} \Big[\frac{e^{A(a \mp x)}}{\sqrt{B - A}} P^{\pm}(iA) \prod_{k=1}^{n} (A + iz_k) \Big| \prod_{k=1}^{m} (A + i\gamma_k) \Big] +$$

$$O(e^{-2Ba})$$

$$N_k^{\pm} = T^{\pm}(z_k) S^{\pm}(z_k) [K_+'(-z_k)]^{-1}, \qquad P^{\pm}(x) = T_0^{\pm}(x) S^{\mp}(-x)$$

$$l = n - m - 1$$

$$M_k^{\pm} = T_0^{\pm}(\gamma_k) S^{\mp}(-\gamma_k) [R_+'(-\gamma_k)]^{-1}, \qquad R_{\pm}(x) = K_{\pm}^{-1}(x)$$

$$S^{\pm}(x) = -e^{\mp ia\eta} R_{\pm}(\eta)(x \mp \eta)^{-1} \pm \frac{1}{2} \sum_{k=1}^{n} a_k^{\pm} (z_k + x)^{-1}$$

$$T^{\pm}(\eta) = \operatorname{erf} \sqrt{(B + i\eta)(a \pm x)}, \qquad T_0^{\pm}(\eta) = \operatorname{erf} \sqrt{(B + i\eta)(-a \pm x)}$$

The degenerate component of the solution is disclosed in the right side, as are also the terms with the singularity characteristic on the stamp edges for contact problems [3] which diminish exponentially with distance from the stamp edges.

Multiplying (4.3) by the time factor $\exp(-i\omega t)$, we detect a wave pair with nondecreasing amplitude under the stamp m_1. The remaining terms in (4.3) diminish exponentially with distance from the stamp edges.

Multiplying (4.4) by $\exp(-i\omega t)$, we note that m waves with nondecreasing amplitude move on the free cylinder surface on both sides of the stamp. The remaining terms in the right side of (4.4) decrease exponentially. As must have been expected, there are no reflected waves here. If the complex coefficients $M_k^{\pm}$ and $N_k^{\pm}$ in (4.4) and (4.3) are represented in trigonometric form, then values of the amplitude and phase shift can be obtained for the appropriate wave.

5. It can be concluded from the above that the clearest wave process appears sufficiently far from the stamp edges. Let us investigate the functions $q(x)$, $\varphi^+(x)$ and $\varphi^-(x)$ under the assumption $|a-|x||>B^{-1}$ in the case of a flat stamp ($\eta=0$). We have

$$q(x)=K^{-1}(0)+\sum_{k=1}^{n}N_k\left[e^{iz_k(a+x)}+e^{iz_k(a-x)}\right]+O\left(e^{-B(a-|x|)}\right) \tag{5.1}$$

$$\varphi^{\pm}(x)=\varphi(\pm x-a),\qquad \pm x>a$$

$$\varphi(t)=\sum_{k=1}^{m}S_k e^{i\gamma_k t}+X(t)+O(e^{-Bt}),\quad t>0$$

$$N_k=R_+(0)\left[\sum_{p=1}^{n}r_p(z_p+z_k)^{-1}\sum_{s=1}^{n}B_{ps}^{\ +}z_s^{-1}-z_k^{-1}\right][K_+'(-z_k)]^{-1}$$

$$S_k=R_+(0)\left[\gamma_k^{-1}-\sum_{s=1}^{n}r_s(\gamma_k-z_s)^{-1}\sum_{p=1}^{n}B_{sp}^{\ +}z_p^{-1}\right][R_+'(-\gamma_k)]^{-1}$$

$$X(x)=\frac{(-1)^{n-m-1}}{(n-m-1)!}\frac{\partial^{n-m-1}}{\partial A^{n-m-1}}\left[\frac{e^{-Ax}}{\sqrt{B-A}}\prod_{k=1}^{n}(A+iz_k)\times\right.$$
$$\left.\prod_{k=1}^{m}(A+i\gamma_k)^{-1}\left[A^{-1}-\sum_{k=1}^{n}r_k(A+iz_k)^{-1}\sum_{s=1}^{n}B_{ks}^{+}z_s^{-1}\right]R_+(0)\right]$$

We conclude from the form of the right side of the first relationship in (5.1) that in this case the wave process under the stamp, at a sufficient distance from the edges, is determined by the sum of m_1 standing waves. The remaining terms if the first relationship in (5.1), except for the degenerate one, decrease sharply since $\operatorname{Im} z_k>0$, $k=m_1+1, m_1+2,\ldots,n$. Also detected are m waves moving on the free cylinder surface on both sides of the stamp. The function $X(x)$ in the fourth relationship in (5.1) is exponentially decreasing.

As an illustration, let us consider the vibration of a stamp in the cases $\varkappa=1.9$ and $\varkappa=4.45$.

We have for $a=7$, $B=10$, $n=m+4$:

1) $A=3.441$ and $K^{-1}(0)=4.057$ with an error less than 7% for $\varkappa=1.9$, $m=m_1=1$.

2) $A=7.774$, $K^{-1}(0)=-1.783$ with less than 2% error for $\varkappa=4.45$. $m=3$, $m_1=2$.

The remaining quantities needed to compute $q(x)$ for $|x|<6$ and $\varphi(x)$ for $x>1$

can be taken from Table 1. As a computation shows, the function $X(x)$ in the third relationship in (5.1) can be neglected on the section $x > 1$.

Table 1

$\varkappa$	$\operatorname{Re} z_k$	$\operatorname{Im} z_k$	$\operatorname{Re} N_k$	$\operatorname{Im} N_k$	γ_k	$\operatorname{Re} S_k$	$\operatorname{Im} S_k$
	1.016	—	−0.669	−0.181			
	—	2.261	−0.220	−0.043			
1.90	0.623	4.188	−0.090	0.073	1.232	−0.137	−0.162
	0.623	4.188	0.057	0.101			
	—	7.875	1.947	0.370			
	2.263	—	−0.984	0.402			
	2.379	—	−0.209	0.090	0.752	0.912	0.127
	—	2.940	1.433	−2.575			
4.45	−1.293	7.683	−0.040	0.101	2.357	20.002	0.003
	1.293	7.683	0.036	0.097			
	—	7.803	—	—	4.496	0.102	0.146
	—	11.000	—	—			

In the first case, the function $q(x)$ is described completely satisfactorily for $|x| < 6$ by the formula

$$q(x) = 4.057 + 1.386 \exp(4.335i) \cos 1.016x$$

with less than 7% maximum error for $|x| = 6$. The function $\varphi(x)$ is described completely satisfactorily by the formula

$$\varphi(x) = 0.212 \exp(4.017\, i + 1.232\, ix)$$

with less than 5% maximum error for $x = 1$. The accuracy of both formulas rises with distance from the stamp edges.

In the second case, it can be shown by using Table 1 and the first and fourth relationships in (5.1) that the wave process under the stamp is determined on the section $|x| < 6$ by the sum of two standing waves, while the wave process on the free cylinder surface is determined for $x > 1$ by the sum of three waves moving on both sides of the stamp. Using the trigonometric representations of the complex numbers N_k and S_k we can obtain both the amplitude and the phase shift for each wave.

The author is grateful to I. I. Vorovich, G. Ia. Popov and V. A. Babeshko for attention to the research and valuable comments.

REFERENCES

1. Babeshko, V. A., On the theory of dynamic contact problems. Dokl. Akad. Nauk SSSR, Vol. 201, № 3, 1971.
2. Babeshko, V. A., On the uniqueness of solutions of integral equations of dynamic contact problems. Dokl. Akad. Nauk SSSR, Vol. 210, № 6, 1973.
3. Babeshko, V. A. and Kalinchuk, V. V., On the interaction of a stiff strap with an elastic cylinder. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, № 3, 1973.
4. Berezin, I. S. and Zhidkov, N. P., Calculation Methods. Vol. 1, "Nauka", Moscow, 1966.
5. Babeshko, V. A. and Kalinchuk, V. V., On an approximate method of solving dynamic contact problems. PMM Vol. 38, № 3, 1973.

Translated by M. D. F.